INVARIANT *-PRODUCTS ON COADJOINT ORBITS AND THE SHAPOVALOV PAIRING

A. ALEKSEEV AND A. LACHOWSKA

Abstract. We give an explicit formula for invariant $*$-products on a wide class of coadjoint orbits. The answer is expressed in terms of the Shapovalov pairing for generalized Verma modules.

1. Introduction

The problem of constructing a $*$-product on a manifold M with given Poisson structure was formulated in [BFFLS]. In the case of symplectic manifolds the existence of $*$-products was established in [DL] and in a more geometric fashion in [F]. For general Poisson manifold the existence of $*$-products was proved in [K]. While general existence results are now available, giving explicit formulas for $*$-products remains a difficult task. The first formula of this type was given by Moyal in [M] in the case of a constant Poisson bi-vector on $\mathbb{R}^n$. Further examples that one can consider are linear Poisson brackets on the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$, and non-degenerate Poisson brackets on coadjoint orbits in $\mathfrak{g}^*$. The $*$-products on $\mathfrak{g}^*$ were constructed in *e.g.* [G]. It is a natural idea to construct $*$-products on coadjoit orbits by restriction from $\mathfrak{g}^*$. However, in [CGR] it was proved that for $\mathfrak{g}$ semisimple a smooth $*$-product on $\mathfrak{g}^*$ does not restrict to coadjoint orbits.

Examples of $*$-products on some simple coadjoint orbits ($C\mathrm{P}^n$ and symmetric spaces) can be found *e.g.* in [BBEW], [T]. We shall concentrate on constructing $*$-products on M *invariant* with respect to the transitive G-action. In the case of $G = \mathrm{GL}(n)$ this problem has been addressed in [DM1]. Invariant $*$-products on minimal nilpotent coadjoint orbits of simple Lie groups were constructed in [ABC] and [AB]. In these works the locality axiom (stating that the $*$-product should be defined by a bi-differential operator) is relaxed. In the case of $\mathfrak{g}$ semisimple constructions of $*$-products on coadjoint orbits of semisimple elements were suggested in [A] using the deformation quantization with separation of variables of [Ka] and in [DM2] using the methods of the category theory.

Our main result is an explicit formula for invariant $*$-products on coadjoint orbits G/H for which the corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ (or their complexifications) fit into a decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ (see Section 2.2 for the precise statement of assumptions). Examples include: the space $\mathbb{R}^{2n}$ with constant nondegenerate Poisson bracket (a coadjoint orbit in the dual to the Heisenberg algebra), coadjoint orbits of semisimple elements in semisimple Lie algebras, as well as infinite dimensional examples such as coadjoint orbits in the dual to the Virasoro algebra. Our construction is motivated by the fusion techniques of [EV].

Date: February 1, 2008.

Acknowledgements. We received a lot of help from S. Parmentier who participated in this project in its early stages. We are grateful to S. Gutt, P. Etingof, V. Rubtsov, D. Sternheimer, X. Tang, B. Tsygan and P. Xu for the interest in our work and for useful discussions. This work was supported in part by the Swiss National Science foundation and by the Erwin Schrödinger Institute for mathematical Physics.

2. Preliminaries

In this Section we formulate the problem of finding invariant $*$-products on homogeneous spaces, and state the assumptions which allow us to solve it in an explicit form.

2.1. **Invariant *-products on homogeneous spaces.** Let M be a manifold. Recall that a $*$-product on M is defined by a formal power series in $\hbar$ with coefficients complex bi-differential operators on M, $B = \sum_{n=0}^{\infty} \hbar^n B_n$ with $B_0 = 1$ and $B_n \in \mathrm{Diff}^2(M)$. The $*$-product is then given by formula, $f * g := fg + \sum_{n=1}^{\infty} \hbar^n B_n(f, g)$ for $f, g \in C^\infty(M)$. The main condition imposed on B is that the $*$-product be associative, $f * (g * h) = (f * g) * h$ for all $f, g, h \in C^\infty(M)$.

Let G be a connected Lie group and let $H \subset G$ be a closed Lie subgroup of G. Denote the corresponding Lie algebras by $\mathfrak{g}$ and $\mathfrak{h}$, respectively. The quotient $M := G/H$ carries a transitive action of G. There is an induced G-action on functions on M, and on differential and poly-differential operators. A $*$-product on M is called invariant if it is defined by an invariant formal bi-differential operator B. That is, all bi-differential operators B_n have to be G-invariant.

Recall that the space of invariant differential operators on $M = G/H$ can be expressed as follows, $\mathrm{Diff}_G(M) = (U\mathfrak{g}/U\mathfrak{g} \cdot \mathfrak{h})^H$. Here $U\mathfrak{g}$ is the universal enveloping algebra of $\mathfrak{g}$, $U\mathfrak{g} \cdot \mathfrak{h}$ is the left ideal generated by $\mathfrak{h} \subset \mathfrak{g} \subset U\mathfrak{g}$. The algebra $U\mathfrak{g}$ carries a natural adjoint action of $H \subset G$, and since $\mathrm{Ad}_H(\mathfrak{h}) = \mathfrak{h}$ this action factors to the quotient $U\mathfrak{g}/U\mathfrak{g} \cdot \mathfrak{h}$. The H-invariant part $(U\mathfrak{g}/U\mathfrak{g} \cdot \mathfrak{h})^H$ has an algebra structure induced by the one of $U\mathfrak{g}$. In a similar fashion, the space of invariant N-differential operators on G/H is given by

$$\mathrm{Diff}_G^N(M) = ((U\mathfrak{g}/U\mathfrak{g} \cdot \mathfrak{h})^{\otimes N})^H,$$

where the invariant part is taken with respect to the diagonal H-action on $(U\mathfrak{g}/U\mathfrak{g} \cdot \mathfrak{h})^{\otimes N}$. Formal bi-differential operators which define invariant $*$-products on M take values in the space

$$B \in ((U\mathfrak{g}/U\mathfrak{g} \cdot \mathfrak{h})^{\otimes 2})^H[[\hbar]].$$

Remark 2.1. As opposed to $\mathrm{Diff}_G(M)$, the space $\mathrm{Diff}_G^N(M)$ for $N > 1$ has no natural algebra structure.

Let $\Delta : U\mathfrak{g} \to U\mathfrak{g} \otimes U\mathfrak{g}$ be the standard coproduct of $U\mathfrak{g}$. That is, Δ is an algebra homomorphism such that $\Delta(x) = 1 \otimes x + x \otimes 1$ for all $x \in \mathfrak{g}$. Let $B, C \in \mathrm{Diff}_G^2(M) = ((U\mathfrak{g}/U\mathfrak{g} \cdot \mathfrak{h})^{\otimes 2})^H$. Then expressions $((\Delta \otimes 1)B)(C \otimes 1)$ and $((1 \otimes \Delta)B)(1 \otimes C)$ define unique elements of $\mathrm{Diff}_G^3(M) = ((U\mathfrak{g}/U\mathfrak{g} \cdot \mathfrak{h})^{\otimes 3})^H$. In more detail, let $\hat{B}, \hat{C} \in U\mathfrak{g} \otimes U\mathfrak{g}$ be representatives of the classes $B, C \in (U\mathfrak{g}/U\mathfrak{g} \cdot \mathfrak{h})^{\otimes 2}$. The classes of $((\Delta \otimes 1)\hat{B})(\hat{C} \otimes 1)$ and $((1 \otimes \Delta)\hat{B})(1 \otimes \hat{C})$ in $(U\mathfrak{g}/U\mathfrak{g} \cdot \mathfrak{h})^{\otimes 3}$ are H-invariant and independent of the choice.

The associativity of an invariant $*$-product defined by a formal bi-differential operator B is expressed by the following equality of invariant formal 3-differential operators,

$$((\Delta \otimes 1)B)\,(B \otimes 1) = ((1 \otimes \Delta)B)\,(1 \otimes B) \tag{1}$$

in $((U\mathfrak{g}/U\mathfrak{g}\cdot\mathfrak{h})^{\otimes 3})^H[[\hbar]]$.

Remark 2.2. If H is a connected Lie group one can replace the condition of Ad_H-invariance of B by an algebraic condition of invariance with respect to the adjoint action of $\mathfrak{h}$. It is also a natural context when the Lie algebra $\mathfrak{g}$ is infinite dimensional and the corresponding Lie group may not be available.

Remark 2.3. In the case of $\mathfrak{h}=0$ equation (1) was considered by Drinfeld in [Dr]. In this reference a family of solutions of (1) was constructed in term of the Campbell-Hausdorff series for the Lie algebra $\mathfrak{g}$.

2.2. **Assumptions on $\mathfrak{g}$ and $\mathfrak{h}$. Examples.** Here we list assumptions imposed on the Lie algebra $\mathfrak{g}$ and its Lie sublagebra $\mathfrak{h}$ which allow to construct explicit solutions of equation (1).

- The Lie algebra $\mathfrak{g}$ is $\mathbb{Z}$-graded, $\mathfrak{g}=\oplus_{i\in\mathbb{Z}}\mathfrak{g}_i$, such that each graded component has finite dimension. The adjoint action of H on $\mathfrak{g}$ preserves this grading. Let $\mathfrak{h}=\mathfrak{g}_0$ and denote $\mathfrak{n}_+=\oplus_{i>0}\mathfrak{g}_i$, $\mathfrak{n}_-=\oplus_{i<0}\mathfrak{g}_i$.
- There exists a character $\chi:\mathfrak{g}_0\to\mathbb{C}$ such that the paring $\mathfrak{n}_+\times\mathfrak{n}_-\to\mathbb{C}$ defined by $u,v\mapsto\chi([u,v]_0)$ for $u\in\mathfrak{n}_+$, $v\in\mathfrak{n}_-$ is nondegenerate. Here $x\mapsto x_0$ denotes the projection onto the zero graded component $\mathfrak{g}_0$. Such a character is called nonsingular.

Remark 2.4. If H is connected, the requirement that H preserves the grading is automatically satisfied.

Remark 2.5. The second assumption implies that $\mathfrak{g}_0\neq 0$.

Example 2.6. Assume that $\mathfrak{g}$ is quadratic. That is, there is a nondegenerate $\mathrm{ad}(\mathfrak{g})$-invariant symmetric bilinear form $Q:\mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ of degree zero. In this case, $\mathfrak{g}_0$ is a finite dimensional quadratic Lie algebra with invariant bilinear form the restriction of Q to $\mathfrak{g}_0$. Let $\mathfrak{z}(\mathfrak{g}_0)$ be the center of $\mathfrak{g}_0$. Every element $z\in\mathfrak{z}(\mathfrak{g}_0)$ defines a character χ_z of $\mathfrak{g}_0$, $\chi_z(x)=Q(z,x)$. An element $z\in\mathfrak{z}(\mathfrak{g}_0)$ is called nonsingular if the map $(u,v)\mapsto\chi_z([u,v]_0)\in\mathbb{C}$ defines a nondegenerate paring between the graded components $\mathfrak{g}_i$ and $\mathfrak{g}_{-i}$ for all i. If $\mathfrak{z}(\mathfrak{g}_0)$ is nonempty and contains a nonsingular element, then the set of such elements is Zariski open in $\mathfrak{z}(\mathfrak{g}_0)$.

Example 2.7. Let $\mathfrak{g}$ be a semisimple complex Lie algebra, $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, $\Pi=\{\alpha_i\}_{i=1}^{\mathrm{rank}(\mathfrak{g})}$ be (some choice of) the set of simple roots. The principal grading on $\mathfrak{g}$ is the unique grading such that $\mathfrak{h}=\mathfrak{g}_0$ and all root vectors corresponding to simple roots have degree 1, $e_{\alpha_i}\in\mathfrak{g}_1$. Any regular character χ of $\mathfrak{h}$ defines a nondegenerate paring $\mathfrak{n}_+\times\mathfrak{n}_-\to\mathbb{C}$.

This construction applies verbatim to any Kac-Moody algebra.

Example 2.8. Let $\mathfrak{g}$ be a semisimple complex Lie algebra, and let $z\in\mathfrak{g}$ be a semisimple element. Let $\mathfrak{g}_z$ be the centralizer of z and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ such that $z\in\mathfrak{h}\subset\mathfrak{g}_z$. Define the unique grading on $\mathfrak{g}$ such that $\mathfrak{g}_0=\mathfrak{g}_z$ and for all simple roots in $S_z:=\{\alpha_i, e_{\alpha_i}\notin\mathfrak{g}_z\}$ one has $e_{\alpha_i}\in\mathfrak{g}_1$. The Lie algebra $\mathfrak{p}_+=\mathfrak{g}_0\oplus\mathfrak{n}_+$ is a parabolic subalgebra of $\mathfrak{g}$ with the Levi subalgebra $\mathfrak{g}_z$, corresponding to the subset $S_z\subset\Pi$. The center $\mathfrak{z}(\mathfrak{g}_0)\subset\mathfrak{h}$ consists of the elements orthogonal to S_z. Denote by R the set of roots of $\mathfrak{g}$ and by $R_z\subset R$ the set of roots of $\mathfrak{g}_z$. Then the nonsingular elements of $\mathfrak{z}(\mathfrak{g}_0)$ that define nondegenerate bilinear paring between $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are those with nonvanishing scalar product with all elements of $R\setminus R_z$. They form a Zariski open subset of $\mathfrak{z}(\mathfrak{g}_0)$. In particular z is a central nonsingular element in $\mathfrak{z}(\mathfrak{g}_0)$.

If z is a regular semisimple element of $\mathfrak{g}$ we return to Example 2.7.

Example 2.9. Let $\mathfrak{g} = H_n$ be the Heisenberg Lie algebra generated by c and by $p_i, q_i, i = 1 \dots n$ with the only nonvanishing Lie brackets $[p_i, q_j] = \delta_{ij} c$. Define the grading by setting $\deg(p_i) = 1, \deg(q_i) = -1$ and $\deg(c) = 0$. Then $\mathfrak{g}_0 = \mathbb{C}c$ and any $\chi \in (\mathfrak{g}_0)^*$ such that $\chi(c) \neq 0$ is a nonsingular character.

Example 2.10. Let $\mathfrak{g} = \mathbf{Vir}$ be the Virasoro algebra with $\mathfrak{g}_n = \mathbb{C}L_n, n \in \mathbb{Z}, n \neq 0$ and $\mathfrak{g}_0 = \mathbb{C}L_0 \oplus \mathbb{C}\mathfrak{c}$ with the Lie bracket $[L_n, L_m] = (n-m)L_{n+m} + \delta_{n+m,0}\frac{n^3-n}{12}\mathfrak{c}$. Let $\chi \in \mathfrak{g}_0^*$ be defined so that $\chi(L_0) = \Delta, \chi(\mathfrak{c}) = c$ for some $\Delta, c \in \mathbb{C}$. Then χ defines the paring $(L_n, L_{-n}) = 2n\Delta + \frac{n^3-n}{12}c$ which is nondegenerate for all pairs $\Delta, c \in \mathbb{C}$ such that $\Delta + \frac{n^2-1}{24}c \neq 0$ for all $n \in \mathbb{Z}_{>0}$.

2.3. **Relation to coadjoint orbits.** Let $\mathfrak{g}^* := \oplus_{i \in \mathbb{Z}} \mathfrak{g}_i^*$ be the (graded) dual to the Lie algebra $\mathfrak{g}$. It carries a natural coadjoint action of $\mathfrak{g}$. An element $\chi \in \mathfrak{g}_0^* \subset \mathfrak{g}^*$ defines a point in $\mathfrak{g}^*$. Our second assumption is equivalent to saying that $\mathfrak{g}_0$ is the coadjoint stabilizer of χ.

Assume that $\mathfrak{g}$ is finite dimensional. Then there is a connected simply connected Lie group G with Lie algebra $\mathfrak{g}$ and with a natural coadjoint action of G on $\mathfrak{g}^*$. Denote the stabilizer of χ by $G_0 \subset G$. It is a closed subgroup of G with Lie algebra $\mathfrak{g}_0$. A coadjoint orbit of χ under the G-action is a homogeneous space $M = G/G_0$. Our construction of invariant $*$-products applies if the adjoint action of G_0 preserves the grading. If G_0 is connected, this condition is satisfied automatically. In particular, this is always the case if $\mathfrak{g}$ is a semisimple Lie algebra.

If G_0 is disconnected let H be the connected component of the unit element. Then our method applies to G/H which covers the coadjoint orbit G/G_0.

Let G be a real Lie group and $\mathfrak{g}$ be the corresponding real Lie algebra. Since the $*$-products are usually defined on the space of complex valued functions, it is sufficient that assumptions of the previous Section be satisfied for the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$. For this reason in the rest of the paper we assume that $\mathfrak{g}$ is a complex Lie algebra.

Recall that the coadjoint orbits carry an invariant symplectic form which is constructed as follows. Identify $T_{eH}(G/H) \cong \mathfrak{n}_- \oplus \mathfrak{n}_+$, and define $\omega(u,v) := -\chi([u,v])$. By assumptions, this bilinear form establishes a duality between $\mathfrak{n}_-$ and $\mathfrak{n}_+$. Let u_i and v_i be a pair of dual bases in $\mathfrak{n}_-$ and $\mathfrak{n}_+$, respectively. Then, the inverse of ω is the invariant Kirillov-Kostant-Souriau Poisson bi-vector on G/H which is equal to $\sum_i u_i \wedge v_i$ at eH. An additional constraint which is often imposed on the bi-differential operator B is that the skew-symmetric part of B_1 be equal a given Poisson bi-vector. In our case, this condition reads

$$B_1 - B_1^t = \sum_i u_i \wedge v_i, \tag{2}$$

where B_1^t is a bi-differential operator obtained by exchanging two copies of $U\mathfrak{g}/U\mathfrak{g} \cdot \mathfrak{h}$ in the tensor product. Geometrically, $\mathfrak{n}_-$ and $\mathfrak{n}_+$ define two distributions in $T(G/H)$. If we deal with a real Lie group $\mathfrak{g}$ and the conditions of the previous Section are satisfied for the real Lie algebra $\mathfrak{g}$, these distributions give rise to transverse Lagrangian polarizations on the orbit. In general, we are getting transverse complex polarizations.

3. Generalized Verma modules and the Shapovalov pairing

In this Section we recall the notion and basic properties of the Shapovalov pairing and of the associated canonical element in the tensor product of two opposite generalized Verma modules. The details on the generalized Verma modules can be found in e.g. [Di].

3.1. **Generalized Verma modules.** Let $\mathfrak{p}_{=} \oplus_{i\geq 0} \mathfrak{g}_i$ and $\mathfrak{p}_- = \oplus_{i\leq 0}\mathfrak{g}_i$. A central character χ of $\mathfrak{g}_0$ can be given a $\mathfrak{p}_\pm$-module structure by letting $\mathfrak{n}_\pm$ act on it by zero. Define the generalized Verma modules by

$$M^\pm = \mathrm{Ind}_{U\mathfrak{p}_\pm}^{U\mathfrak{g}}\chi \cong U(\mathfrak{g}) \otimes_{U\mathfrak{p}_\pm} \chi.$$

In Example 2.7 we recover the Verma module over $\mathfrak{g}$ of highest weight χ, in Example 2.8 - the scalar generalized Verma module induced from the parabolic $\mathfrak{p}_+$. By general properties of the induction, M^+ is isomorphic to $U\mathfrak{n}_-$ as a $U\mathfrak{n}_-$-module generated by $v_\chi = 1 \otimes \chi$. Similarly, $M^- \cong U\mathfrak{n}_+$ as a $U\mathfrak{n}_+$-module generated by v_χ. Both Verma modules inherit natural gradings from $U\mathfrak{n}_+, U\mathfrak{n}_-$.

Suppose that M is a $\mathbb{Z}$-graded $U(\mathfrak{g})$-module, and V is any $U(\mathfrak{g})$-module. Then we define the completed tensor product $M\check{\otimes}V \equiv \oplus_{i\in\mathbb{Z}} M_i \otimes V$ as a $\mathbb{Z}$-graded $U(\mathfrak{g})$-module where elements of $U(\mathfrak{g})$ act by comultiplication. In particular, if both M and N are $\mathbb{Z}$-graded $U(\mathfrak{g})$-modules, then $M\check{\otimes}N$ has two $\mathbb{Z}$-gradings. Often it is convenient to preserve both gradings, $M\check{\otimes}N \equiv \oplus_{i,j} M_i \otimes N_j$, where the elements may have infinite length.

Fix a nonsingular character χ of $\mathfrak{g}_0$. For any $\lambda \in \mathbb{C}$ we consider a rescaled character $\chi_\lambda := \lambda\chi$, and define a paring between $U\mathfrak{n}_-$ and $U\mathfrak{n}_+$ which depends on the parameter λ. Write $U(\mathfrak{g}) \cong U(\mathfrak{g}_0) \oplus (\mathfrak{n}_- U(\mathfrak{g}) \oplus U(\mathfrak{g})\mathfrak{n}_+)$ and let $\phi : U(\mathfrak{g}) \to U(\mathfrak{g}_0)$ be the projection of an element in $U(\mathfrak{g})$ to the first summand along the second. For any $x \in U\mathfrak{n}_-, y \in U\mathfrak{n}_+$, we set

$$(x, y)_\lambda := \chi_\lambda(\phi(S(y)x)),$$

where $S : U\mathfrak{g} \to U\mathfrak{g}$ is the antipode of $U\mathfrak{g}$. That is, S is the unique anti-automorphism of $U\mathfrak{g}$ such that $S(x) = -x$ for all $x \in \mathfrak{g}$.

Let $M^+_\lambda \cong \mathrm{Ind}^{U\mathfrak{g}}_{U\mathfrak{p}_+}\chi_\lambda$ and $M^-_{-\lambda} \cong \mathrm{Ind}^{U\mathfrak{g}}_{U\mathfrak{p}_-}\chi_{-\lambda}$ be the generalized Verma modules. The pairing $(\cdot,\cdot)_\lambda$ gives rise to a pairing between M^+_λ and $M^-_{-\lambda}$. Namely, fix generating vectors $v_\lambda = 1 \otimes \chi_\lambda \in M^+_\lambda$ and $v_{-\lambda} = 1 \otimes \chi_{-\lambda} \in M^-_{-\lambda}$ and let

$$(xv_\lambda, yv_{-\lambda}) := (x, y)_\lambda.$$

This pairing is called the Shapovalov pairing between the generalized Verma modules. It is $U\mathfrak{g}$-invariant in the following sense: $(au, v) = (u, S(a)v)$ for $u \in M^+_\lambda$ and $v \in M^-_{-\lambda}$. The modules M^+_λ and $M^-_{-\lambda}$ are irreducible if and only if $(\cdot,\cdot)_\lambda : U\mathfrak{n}_+ \times U\mathfrak{n}_- \to \mathbb{C}$ is nondegenerate. Indeed, if $x \cdot v_\lambda$ lies in a proper submodule of M^+_λ and $x \in U\mathfrak{n}_-$ has a maximal degree in this submodule, then $(x, y)_\lambda = 0$ for all $y \in U\mathfrak{n}_+$ by the $U(\mathfrak{g})$-invariance of the paring, and conversely.

Proposition 3.1. *Let χ be a nonsingular character of $\mathfrak{g}_0$. Then the pairing $(\cdot,\cdot)_\lambda : U\mathfrak{n}_- \otimes U\mathfrak{n}_+ \to \mathbb{C}$ is nonsingular for almost all $\lambda \in \mathbb{C}$.*

Proof. Let $\{u_i\}$ be a homogeneous basis in $\mathfrak{n}_-$, χ a nonsingular character, and $\{v_i\}$ the dual basis in $\mathfrak{n}_+$ with respect to the paring $(u, v) = -\chi([u, v]_0)$. Choose an order of the elements u_i in each graded component of $\mathfrak{n}_+$, and enumerate the set $\{u_i\}$ by increased grading. Then we have a PBW-type basis in $U(\mathfrak{n}_\pm)$. In particular, in each graded component $(U\mathfrak{n}_-)_{-n}, n \in \mathbb{Z}_+$, there is a basis $\{x^{(n)}_k\}^N_{k=1}$ of monomials in $\{u_i\}$. It can be ordered so that the number of factors in $x^{(n)}_i$ is greater or equal to that in $x^{(n)}_j$ whenever $i \leq j$. For each element $x^{(n)}_k = u^{s_1}_{k_1} u^{s_2}_{k_2} \dots u^{s_r}_{k_r}$ set $y^{(n)}_k = v^{s_1}_{k_1} v^{s_2}_{k_2} \dots v^{s_r}_{k_r}$. The elements $\{y^{(n)}_k\}$ form a basis in $(U\mathfrak{n}_+)_n$. The number $d_k = \sum^r_{i=1} s_i$

will be called the length of $x_k^{(n)}$. Let $\lambda \in \mathbb{C}$ and consider the $N \times N$ matrix $M^n(\lambda) = (x_k^{(n)}, y_l^{(n)})_\lambda$. Its elements are polynomials in λ. The following statements are easy to check:

(a) The order of the polynomial $(x_k^{(n)}, y_l^{(n)})_\lambda$ cannot exceed the length of the shortest of the two monomials $x_k^{(n)}, y_l^{(n)}$.

(b) For two monomials $x_k^{(n)}, y_l^{(n)}$ of the same length d_k, $(x_k^{(n)}, y_l^{(n)})_\lambda$ is a polynomial of order strictly less than d_k unless $l = k$. If $l = k$ then $x_k^{(n)}$ and $y_k^{(n)}$ have the same number of factors which are dual to each other with respect to the $(\cdot, \cdot)$ paring.

(c) We have

$$(x_k, y_k)_\lambda = (\prod_{i=1}^r (s_i)!)\lambda^{d_k} + P_k(\lambda),$$

where P_k is a polynomial of order less than d_k.

The above implies that we can write

$$M^n(\lambda) = D^n(\lambda)C(1 + O(1/\lambda)), \tag{3}$$

where $D^n(\lambda)$ is a diagonal matrix with $[D^n(\lambda)]_{kk} = (\prod_{i=1}^r (s_i)!)\lambda^{d_k}$, the matrix C is constant lower triangular with units on the diagonal, and $O(1/\lambda)$ is a matrix whose entries are polynomials in $1/\lambda$ without a constant term. The determinant of $M^n(\lambda)$ is a polynomial in λ of order $\sum_{k=1}^N d_k$ with a nonzero leading coefficient. Therefore, the matrix $M^n(\lambda)$ is invertible for all but a finite number of values of $\lambda \in \mathbb{C}$. The union of zeros of $M^n(\lambda)$ for all n is a countable subset of $\mathbb{C}$. This completes the proof. $\square$

3.2. **Canonical element F_λ.** Let $\lambda \in \mathbb{C}$ such that the pairing $(\cdot, \cdot)_\lambda$ be nonsingular. Denote by $F_\lambda \in U\mathfrak{n}_- \check{\otimes} U\mathfrak{n}_+$ the canonical element corresponding to the pairing. It has the following form, $F_\lambda \in 1 + (U\mathfrak{n}_-)_{<0} \check{\otimes} (U\mathfrak{n}_+)_{>0}$. Let $M_\lambda^+ \check{\otimes} M_{-\lambda}^-$ be a completed tensor product of two irreducible generalized Verma modules with generating vectors v_λ and $v_{-\lambda}$. Then $F_\lambda(v_\lambda \otimes v_{-\lambda})$ is the canonical element with respect to the Shapovalov pairing. In particular, it is $U\mathfrak{g}$-invariant. Choose a generating vector $v \in V_0$ in the trivial $U\mathfrak{g}$ module $V_0 \cong \mathbb{C}$. Then $v \mapsto F_\lambda(v_\lambda \otimes v_{-\lambda})$ defines a $U\mathfrak{g}$-homomorphism $V_0 \to M_\lambda^+ \check{\otimes} M_{-\lambda}^-$.

Proposition 3.2. *The element F_λ is a meromorphic function of λ which is holomorphic at $\lambda = \infty$. The residue of F_λ at $\lambda = 0$ is given by formula,*

$$\mathrm{Res}_{\lambda=0} F_\lambda = \sum_{i=1}^n u_i \otimes v_i \in \mathfrak{n}_- \check{\otimes} \mathfrak{n}_+ \subset U\mathfrak{n}_- \check{\otimes} U\mathfrak{n}_+. \tag{4}$$

Proof. Using bases $\{x_k^{(n)}\}$ and $\{y_k^{(n)}\}$ of Proposition 3.1 one can write the element F_λ in the following form,

$$F_\lambda = \sum_{n=0}^\infty \sum_{kl} (M^n(\lambda))_{lk}^{-1} \, x_k^{(n)} \otimes y_l^{(n)}.$$

Decomposition (3) implies that matrix elements $(M^n(\lambda))_{lk}^{-1}$ are rational functions of λ holomorphic at $\lambda = \infty$. Hence, each bi-graded component of F_λ is a rational function of λ holomorphic at $\lambda = \infty$, which amounts to saying that F_λ is meromorphic and regular at $\lambda = \infty$.

To compute the residue of F_λ note that for the matrix C of (3) one has $C_{kl} = \delta_{kl}$ for k, l with $d_k = d_l = 1$. This implies the same property for the inverse matrix, $C^{-1}_{kl} = \delta_{kl}$ for k, l with $d_k = d_l = 1$. We write,

$$\mathrm{Res}_{\lambda=0}\,(M^n(\lambda))^{-1}_{lk} = \mathrm{Res}_{\lambda=0} \sum_m (1+O(\lambda^{-1}))_{lm} C^{-1}_{mk} \lambda^{-d_k} (\prod_i s_i!)^{-1} = \mathrm{Res}_{\lambda=0}\, C^{-1}_{lk} \lambda^{-d_k} (\prod_i s_i!)^{-1}$$

This formula shows that the residue vanishes unless $d_k = 1$. Then, since C^{-1} is lower triangular with $d_l \leq d_k$ for $l \geq k$ one has $d_l = 1$ as well. Hence, the only matrix elements which have nonvanishing residues are diagonal entries $k = l$ with $d_k = 1$, in which case the residue is equal to one. Since the elements with $d = 1$ are basis elements in $\mathfrak{n}_-$ and $\mathfrak{n}_+$ we obtain (4) as required. □

Remark 3.3. Formula (3) implies that the coefficient $(M^n(\lambda))^{-1}_{lk}$ in front of the element $x^{(n)}_k \otimes y^{(n)}_l$ in F_λ is a rational function in λ of order less or equal to $-d_k$. By exchanging the roles of k and l we see that in fact the order is $\leq -\max(d_k, d_l)$.

Remark 3.4. In case when $\mathfrak{g}$ is finite dimensional, let $d_{max} \in \mathbb{Z}_+$ be such that $(\mathfrak{n}_+)_l = 0$ for all $l > d_{max}$. Then we have the following estimate for the length d_k of monomials in the basis of $(U\mathfrak{n}_+)_n$: $d_k \geq [\frac{n}{d_{max}}]$. Therefore in the Taylor series in $1/\lambda$ for the coefficients of the matrix $(M^n(\lambda))^{-1}$ the leading term is of order greater or equal to $[\frac{n}{d_{max}}]$. The coefficients of a given order m in $1/\lambda$ can appear only in matrices $(M^n(\lambda))^{-1}$ with $n \leq m d_{max}$. Since each graded component of $U\mathfrak{n}_+$ is finite dimensional, F_λ has only finitely many bi-graded components with coefficients of a given order in $1/\lambda$.

4. Solutions of the associativity equation

In this Section we relate the Shapovalov pairing with the associativity equation (1) for the pair $\mathfrak{g}, \mathfrak{h} := \mathfrak{g}_0$. In the proof of the associativity equation we shall use $U\mathfrak{g}$-homomorphisms from generalized Verma modules to various completed tensor products. Our argument uses the method introduced in [EV]; detailed exposition can be found in [ES].

4.1. $U\mathfrak{g}$-homomorphisms of generalized Verma modules. One useful property of $U\mathfrak{g}$-homomorphisms is given in the following Proposition.

Proposition 4.1. *Let M^+_λ be an irreducible generalized Verma module with generating vector v_λ. Suppose that for a $U\mathfrak{g}$-module $\mathcal{V}$, there exists a $U\mathfrak{g}$-invariant element $z \in M^+_\lambda \check{\otimes} \mathcal{V}$ such that $z \in v_\lambda \otimes w + (Un_-)_{<0} \cdot v_\lambda \check{\otimes} \mathcal{V}$ for a certain $w \in \mathcal{V}$. Then such element is unique.*

Proof. Assuming that an invariant element exists, we will construct it iductively starting from the summand $v_\lambda \otimes w$. Let $\{y_k\}$ and $\{x_k\}$ be the bases in $U\mathfrak{n}_+$ and $U\mathfrak{n}_-$ which consist of the elements of the bases in each graded component of $U\mathfrak{n}_+$ and $U\mathfrak{n}_-$ constructed in the proof of Proposition 3.1, and ordered by the increased grading. Then we can write $z = v_\lambda \otimes w + \sum_{k>0} x_k v_\lambda \otimes w_k$ for some elements $w_k \in \mathcal{V}$. By assumptions we have $\Delta(y_k) z = 0$ for all $k > 0$. Let y_1 be a basis element of degree 1. Then $\Delta(y_1) z = v_\lambda \otimes y_1 w + \sum_k y_1 x_k v_\lambda \otimes w_k + \sum_k x_k v_\lambda \otimes y_1 w_k$. By construction of the basis, there is only one x_s (of degree 1) such that $y_1 x_s v_\lambda = v_\lambda$, and therefore one determines uniquely the element $w_s \in \mathcal{V}$. Then proceed by induction on the degree of y_k. □

Remark 4.2. In case when $\mathcal{V} = V$ is a finite dimensional module, $\check{\otimes}$ is the usual tensor product, the statement follows from the Frobenius reciprocity of the induction:

$$\mathrm{Hom}_{U\mathfrak{g}}(V_0, M_\lambda^+ \otimes V) = \mathrm{Hom}_{U\mathfrak{g}}(M_{-\lambda}^-, V) = \mathrm{Hom}_{U\mathfrak{p}_-}(\chi_{-\lambda}, V).$$

Therefore, the space of $U\mathfrak{g}$-homomorphisms $V_0 \to M_\lambda^+ \otimes V$ coincides with the space of $(U\mathfrak{n}_-)$-invariant vectors in V with the action of $U\mathfrak{g}_0$ given by the character $\chi_{-\lambda}$.

Proposition 4.3. *Let $A : \mathfrak{g} \to \mathfrak{g}$ be a Lie algebra automorphism of $\mathfrak{g}$ preserving the grading. Then the element F_λ is invariant with respect to the natural action of A on $U\mathfrak{n}_-\check{\otimes}U\mathfrak{n}_+$.*

Proof. Since A preserves the grading, the element $A(F_\lambda)(v_\lambda \otimes v_{-\lambda}) \in M_\lambda^+\check{\otimes}M_{-\lambda}^-$ is of the form, $v_\lambda \otimes v_{-\lambda} + (U\mathfrak{n}_-)_{<0} \cdot v_\lambda \otimes (U\mathfrak{n}_+)_{>0} \cdot v_{-\lambda}$. The map A being a Lie algebra automorphism, $A(F_\lambda)(v_\lambda \otimes v_{-\lambda})$ is $U\mathfrak{g}$-invariant. Hence, by Proposition 4.1 one has $A(F_\lambda)(v_\lambda \otimes v_{-\lambda}) = F_\lambda(v_\lambda \otimes v_{-\lambda})$ which implies $A(F_\lambda) = F_\lambda$. □

Remark 4.4. Assume that $\mathfrak{g}$ integrates to a Lie group G, and $H \subset G$ is a subgroup with Lie algebra $\mathfrak{g}_0$ such that Ad_H preserves the grading. Then, the element F_λ is Ad_H-invariant.

Proposition 4.5. *Let $\lambda \in \mathbb{C}$ be such that the module M_λ^+ is irreducible and denote $p : U\mathfrak{g} \to U\mathfrak{g}/U\mathfrak{g} \cdot \mathfrak{g}_0$ the natural projection. Then*

$$(1 \otimes p \otimes 1)\left[(\Delta \otimes 1)F_\lambda(F_\lambda \otimes 1)\right] = (1 \otimes p \otimes 1)\left[(1 \otimes \Delta)F_\lambda(1 \otimes F_\lambda)\right] \tag{5}$$

in $U\mathfrak{n}_-\check{\otimes}U\mathfrak{g}/U\mathfrak{g} \cdot \mathfrak{g}_0\check{\otimes}U\mathfrak{n}_+$.

Remark 4.6. In case when $\mathfrak{g}$ is semisimple and $\mathfrak{g}_0$ its Cartan subalgebra, the equation (5) is a projection from $U(\mathfrak{g})^{\check{\otimes}3}$ to $U\mathfrak{n}_-\check{\otimes}U\mathfrak{g}/U\mathfrak{g} \cdot \mathfrak{g}_0\check{\otimes}U\mathfrak{n}_+$ of the dynamical cocycle condition which was considered in [EV], [EE] and [ESS]. The element satisfying such a condition is called a dynamical twist and is used furter to construct solutions of the dynamical Yang-Baxter equation.

For V_0 the trivial representation of $U\mathfrak{g}$ define $\mathcal{V}_0 = \mathrm{Ind}_{U(\mathfrak{g}_0)}^{U(\mathfrak{g})} V_0$. Choose a nonzero vector $v \in V_0$ and denote $\tilde{v} := 1 \otimes v$ the generating vector in $\mathcal{V}_0$.

Lemma 4.7. *Let $\lambda \in \mathbb{C}$ be such that the module M_λ^+ is irreducible.*

(a) *There exists a unique $U(\mathfrak{g})$-homomorphism*

$$\mathcal{F}_\lambda^+ : M_\lambda^+ \to M_\lambda^+\check{\otimes}\mathcal{V}_0$$

such that

$$\mathcal{F}_\lambda^+(v_\lambda) \in v_\lambda \otimes \tilde{v} + (U\mathfrak{n}_-)_{<0} \cdot v_\lambda \otimes U\mathfrak{g} \cdot \tilde{v}.$$

Explicitly,

$$\mathcal{F}_\lambda^+(v_\lambda) = F_\lambda(v_\lambda \otimes \tilde{v}).$$

(b) *Similarly, there exists a unique $U\mathfrak{g}$-homomorphism*

$$\mathcal{F}_{-\lambda}^- : M_{-\lambda}^- \to \mathcal{V}_0\check{\otimes}M_{-\lambda}^-$$

such that for a fixed vector $v \in V_0$

$$\mathcal{F}_\lambda^-(v_{-\lambda}) \in \tilde{v} \otimes v_{-\lambda} + U\mathfrak{g} \cdot \tilde{v} \otimes (U\mathfrak{n}_+)_{>0} \cdot v_{-\lambda}.$$

Explicitly,

$$\mathcal{F}_{-\lambda}^-(v_{-\lambda}) = F_\lambda(\tilde{v} \otimes v_{-\lambda}).$$

Proof. Consider the first statement. The element $F_\lambda(v_\lambda \otimes \tilde{v})$ is $U\mathfrak{n}_+$-invariant because of the $U\mathfrak{g}$-invariance of the Shapovalov's paring and the $U\mathfrak{n}_+$-invariance of v_λ. The subalgebra $U\mathfrak{g}_0$ acts on $F_\lambda(v_\lambda \otimes \tilde{v})$ by the character χ_λ. Therefore, by the universal property of the generalized Verma module, there is a homomorphism $\mathcal{F}^+_\lambda : M^+_\lambda \to M^+_\lambda \check{\otimes} \mathcal{V}_0$ mapping the generating vector v_λ to $F_\lambda(v_\lambda \otimes \tilde{v})$. The proof of the uniqueness for a given choice of $v \in V_0$ coincides verbatim with the proof of Proposition 4.1. The second statement is proved similarly. □

Lemma 4.8. *Let $\lambda \in \mathbb{C}$ be such that M^+_λ is irreducible. The two morphisms of $U\mathfrak{g}$-modules*

$$V_0 \to M^+_\lambda \check{\otimes} M^-_{-\lambda} \xrightarrow{\mathcal{F}^+_\lambda \check{\otimes} \mathrm{id}} M^+_\lambda \check{\otimes} \mathcal{V}_0 \otimes M^-_{-\lambda}$$

and

$$V_0 \to M^+_\lambda \check{\otimes} M^-_{-\lambda} \xrightarrow{\mathrm{id} \otimes \mathcal{F}^-_\lambda} M^+_\lambda \check{\otimes} \mathcal{V}_0 \check{\otimes} M^-_{-\lambda}$$

coincide.

Proof. Both homomorphisms map v to $v_\lambda \otimes \tilde{v} \otimes v_{-\lambda} + (U\mathfrak{n}_-)_{<0} \cdot v_\lambda \check{\otimes} U(\mathfrak{g}) \cdot \tilde{v} \check{\otimes} (U\mathfrak{n}_+)_{>0} \cdot v_{-\lambda}$. Such a homomorphism is unique by Lemma 4.7. □

Proof of Proposition 4.5. Written in terms of $F_\lambda \in \sum_i (U\mathfrak{n}_+)_{-i} \otimes (U\mathfrak{n}_-)_i$, the equality of the two homomorphisms in Lemma 4.8 gives the statement of the theorem. □

4.2. **Invariant $*$-products.** Denote by $\pi : U\mathfrak{g}^{\otimes 2} \to (U\mathfrak{g}/U\mathfrak{g} \cdot \mathfrak{g}_0)^{\otimes 2}$ the natural projection. The following theorem is the main result of this paper.

Theorem 4.9. *Let χ be a nonsingular character of $\mathfrak{g}_0$ and F_λ be the corresponding canonical element in $U\mathfrak{n}_- \check{\otimes} U\mathfrak{n}_+$. Then, the element $B := \pi(F_{\hbar^{-1}})$ takes values in $((U\mathfrak{g}/U\mathfrak{g} \cdot \mathfrak{g}_0)^{\otimes 2})^{\mathfrak{g}_0}[[\hbar]]$ and satisfies the associativity equation* (1)*.*

Proof. Recall that F_λ is a meromorphic function of λ holomorphic at $\lambda = \infty$. Hence, $F_{\mu^{-1}}$ is holomorphic at zero and defines a formal power series $F_{\hbar^{-1}} \in (U\mathfrak{n}_- \check{\otimes} U\mathfrak{n}_+)[[\hbar]]$ with projection $B := \pi(F_{\hbar^{-1}}) \in (U\mathfrak{g}/U\mathfrak{g} \cdot \mathfrak{g}_0)^{\otimes 2}[[\hbar]]$. The $U\mathfrak{g}$-invariance of the Shapovalov pairing implies the $\mathfrak{g}_0$-invariance of $\pi(F_\lambda)$ and as a consequence the $\mathfrak{g}_0$-invariance of B.

Projecting equation (5) to $(U\mathfrak{g}/U\mathfrak{g} \cdot \mathfrak{g}_0)^{\otimes 3}$ yields the associativity equation for $\pi(F_\lambda)$. Since the latter is holomorphic at infinity, the associativity equation for B follows. □

Remark 4.10. As usual, assume that $\mathfrak{g}$ integrates to a Lie group G and $\mathfrak{g}_0$ integrates to a subgroup $H \subset G$ such that Ad_H preserves the grading. Then $\pi(F_\lambda)$ is an element in $((U\mathfrak{g}/U\mathfrak{g} \cdot \mathfrak{g}_0)^{\otimes 2})^H[[\hbar]]$ which satisfies the associativity equation. Hence, it defines an invariant $*$-product on G/H.

Remark 4.11. In a recent paper [DM2], a categorical approach to constructing dynamical twists is developed. It allows the authors to obtain a quantization of function algebras on semisimple coadjoint orbits for $\mathfrak{g}$ reductive and $\mathfrak{g}_0$ its Levi subalgebra. The difference between the approach of [DM2] and the one chosen in the present paper is that we do not use finite dimensional representations of $U\mathfrak{g}$ and the harmonic analysis on G/H.

Proposition 4.12. *Let $\{u_i\}$ and $\{v_i\}$ be dual bases in $\mathfrak{n}_-$ and $\mathfrak{n}_+$ with respect to the pairing $-\chi([\cdot,\cdot]_0)$. Then the element B has the form*

$$B = 1 + \hbar\pi(\sum_i u_i \otimes v_i) + O(\hbar^2).$$

Proof. By Proposition 3.2 the first two terms in the Taylor expansion of F_λ at $\lambda = \infty$ are as follows,

$$F_\lambda = 1 + \lambda^{-1} \sum_i u_i \otimes v_i + O(\lambda^{-2}).$$

Replacing $\lambda^{-1} \mapsto \hbar$ and applying projection π yields the result. □

Remark 4.13. The skew-symmetric part of the first order term in $\hbar$ in the Taylor expansion of B defines an invariant bi-vector on $M = G/H$. Its value at $eH \in M$ is given by $\sum_{i=1}^n u_i \wedge v_i$ which is exactly the Kirillov-Kostant-Souriau bi-vector on the coadjoint orbit.

Remark 4.14. By Remark 3.3 the elements $x_k \otimes y_l$ which occur with a factor λ^{-n} in the element F_λ have the property $d_k, d_l \leq n$. Hence, the bi-differential operators B_n in the $*$-product $\pi(F_{\hbar^{-1}})$ have the order $\leq n$ in each factor. Such $*$-products are called *natural.* In [GR] it is shown that a natural $*$-product induces a symplectic connection on the underlying manifold M. In the case of coadjoint orbits considered in this paper such a connection is defined by a $\mathfrak{g}_0$-invariant complement to to $\mathfrak{g}_0$ in $\mathfrak{g}$, $\mathfrak{n}_+ \oplus \mathfrak{n}_- \subset \mathfrak{g}$.

Example 4.15. Let $\mathfrak{g} = H_n$ be the Heisenberg algebra defined in Example 2.9, and let $\chi : c \mapsto w \in \mathbb{C}$ be a nonsingular character, $w \neq 0$. Then the matrix of the Shapovalov paring in each graded component $(\mathrm{U}(n_-)_{-k}, \mathrm{U}(n_+)_k)$ is diagonal, with matrix elements

$$(q_1^{k_1} \dots q_n^{k_n}, p_1^{k_1} \dots p_n^{k_n})_\lambda = (-\lambda w)^k \prod_{i=1}^n k_i!,$$

where $k = k_1 + \cdots + k_n$. The corresponding inverse element F_λ is given by

$$F_\lambda = \sum_{k_1, \dots k_n = 0}^{\infty} \frac{(-1)^k}{(\lambda w)^k k_1! \dots k_n!} q_1^{k_1} \dots q_n^{k_n} \otimes p_1^{k_1} \dots p_n^{k_n}.$$

Using the product structure of $U\mathfrak{n}_- \otimes U\mathfrak{n}_+$ one can write the answer for B in the following compact form,

$$B = \exp\left(-\frac{\hbar}{w} \sum_{i=1}^n q_i \otimes p_i \right).$$

This gives a $*$-product on the coadjoint orbit $G/G_0 = \mathbb{R}^{2n}$ of the Heisenberg group acting on $\mathfrak{g}^*$. It is a ‘normal ordered’ version of the Moyal product [M].

Example 4.16. Let $\mathfrak{g} = sl(2, \mathbb{C})$ with generators e, f, h and the Lie brackets $[e, f] = h, [h, e] = 2e, [h, f] = -2f$. Then $\mathfrak{g}_0 = \mathbb{C}h$ and $\chi(h) = z \in \mathbb{C}$, $z \neq 0$ defines a nonsingular character. The element B associated to $sl(2, \mathbb{C})$ and χ is given by the formula

$$B = \sum_{n=1}^{\infty} \hbar^n \left[\frac{(-1)^n}{n! z(z - \hbar) \dots (z - (n-1)\hbar)} \right] f^n \otimes e^n. \tag{6}$$

Remark 4.17. By the theorem of Cahen, Gutt and Rawnsley [CGR], in case when $\mathfrak{g}$ is semisimple, there are no $*$-products on $\mathfrak{g}^*$ with the first order term given by the KKS Poisson bi-vector which restrict to the coadjoint orbits. This is illustrated by the above example: starting from the second graded component, the expression (6) for the $*$-product is singular at $z = 0$.

Remark 4.18. Set $G = SU(2)$. Then the stabilizer of a nonsingular character $\chi \in \mathfrak{g}^*$ is isomorphic to $G_0 = U(1)$. The homogeneous space $M = G/G_0 = S^2$ is the underlying manifold for the $*$-product constructed above. In this case explicit formula for the $*$-product similar to (6) appeared before in physics literature in [P], [N] and [HNT].

Example 4.19. Let **Vir** be the Virasoro algebra defined in Example 2.10 and χ a nonsingular character. We will use the notation of 2.10 to give an explicit expression for the element B, corresponding to **Vir** and χ, up to the second graded component. The character χ being nonsingular implies in particular that $\Delta \neq 0$ and $A := -32\Delta^3 - 4\Delta^2 c \neq 0$. Denote $B := 20\Delta^2 - 2\Delta c$, and set $D = A + \hbar B$. Then

$$B = 1 - \hbar\left[\tfrac{1}{2\Delta}\right] L_{-1} \otimes L_1 + \hbar\left[\tfrac{8\Delta^2}{D} + \hbar\tfrac{4\Delta}{D}\right] L_{-2} \otimes L_2 + \hbar^2\left[\tfrac{6\Delta}{D}\right] L_{-2} \otimes L_1^2 -$$
$$-\hbar^2\left[\tfrac{6\Delta}{D}\right] L_{-1}^2 \otimes L_2 + \hbar^2\left[\tfrac{A}{8D}\right] L_{-1}^2 \otimes L_1^2 + (U\mathfrak{n}_-)_{\leq -3} \otimes (U\mathfrak{n}_+)_{\geq 3}.$$

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University of Geneva, Section of Mathematics, 2-4 rue du Lièvre, 1211 Genève 24, Switzerland
E-mail address: `alekseev@math.unige.ch`

Department of Mathematics, M.I.T., 77 Massachusetts ave., Cambridge MA 02139, USA
E-mail address: `lachowska@math.mit.edu`